

Nonequilibrium statistical operator method in the Renyi statistics

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Abstract

The generalization of the Zubarev nonequilibrium statistical operator (NSO) method for the case of Renyi statistics is proposed when the relevant statistical operator (or distribution function) is obtained based on the principle of maximum for the Renyi entropy. The nonequilibrium statistical operator and corresponding generalized transport equations for the reduced-description parameters are obtained. A consistent description of kinetic and hydrodynamic processes in the system of interacting particles is considered as an example.

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I. INTRODUCTION

In investigations of complex self-organizing, fractal structures and various physical phenomena such as subdiffusion, turbulence, chemical reactions as well as various economical, social and biological systems Gibbs distribution function does not provide agreement with observable phenomena. For these systems the power distributions are inherent [1]. They can not be obtained from the maximum entropy principle for the Boltzmann-Gibbs entropy underlying both an equilibrium and nonequilibrium statistical thermodynamics [2–4].

In papers of A.G. Bashkirov [5–8] the use of the Renyi [9, 10] entropy as statistical entropy for investigation of complex systems is proposed. It depends on the parameter q ($0 < q \leq 1$) and at $q = 1$ it coincides with Boltzmann-Gibbs entropy. Based on the maximum entropy principle for the Renyi entropy the power Renyi distribution was obtained in the case of equilibrium. At $q = 1$ it develops into the Gibbs canonical distribution. Herewith $\eta = 1 - q$ is considered as an order parameter. With the increase of it the statistical Renyi entropy grows to its maximum which the power Renyi distribution corresponds to. Moreover the Renyi entropy derivative with respect to η suffers a sudden change at $\eta = 0$, that is a sort of phase transition to more ordered equilibrium state takes place. The papers of Abe [11, 12], Arimitsu [13, 14], Lesche [15], Masi [16] and others were devoted to axiomatic argumentation and problems of stability of the Renyi entropy and its linearized form — the Havrda-Charvat-Tsallis entropy [17, 18]. The properties of the Renyi entropy are discussed in books [9, 19, 20] as well. Nowadays the Tsallis entropy is widely applied in various directions of nonextensive statistical mechanics [21–23]. The examples are the phenomena of subdiffusion [24] and turbulence [25, 26], the investigations of transport coefficients in gases and plasma [27, 28] as well as quantum dissipative systems [29]. The problems of the construction of equilibrium thermodynamics in the framework of the Renyi and Tsallis statistics were discussed in [30–37]. The energy fluctuations [33], kinetics of nonequilibrium plasma [38], problems of self-gravitating systems [39], ozone layer [40], universality in non-Debye relaxation [41] and complex systems [42, 43] were investigated within the Tsallis statistics.

In the present paper one approach of formulation of an extensive statistical mechanics of nonequilibrium processes is considered based on the nonequilibrium statistical operator method by Zubarev [2–4] and the maximum entropy principle for the Renyi entropy. The

consistent description of kinetic and hydrodynamic processes in the system of classical interacting particles is considered as an example.

II. MAXIMUM ENTROPY PRINCIPLE FOR THE RENYI ENTROPY AND NONEQUILIBRIUM STATISTICAL OPERATOR

Nonequilibrium state of a classical or quantum system of interacting particles is completely described by the nonequilibrium statistical operator (nonequilibrium distribution function) $\varrho(x^N; t)$. The latter satisfies the classical or quantum Liouville equation

$$\frac{\partial}{\partial t}\varrho(x^N; t) + iL_N\varrho(x^N; t) = 0. \quad (1)$$

iL_N is the Liouville operator the system of interacting particles which in classical case has the following form:

$$iL_N = \sum_{l=1}^N \frac{\vec{p}_l}{m} \cdot \frac{\partial}{\partial \vec{r}_l} - \frac{1}{2} \sum_{l \neq j=1}^N \frac{\partial}{\partial \vec{r}_l} \Phi(r_{lj}) \left(\frac{\partial}{\partial \vec{p}_l} - \frac{\partial}{\partial \vec{p}_j} \right). \quad (2)$$

Here $x_j = \{\vec{p}_j, \vec{r}_j\}$ is the phase variables of j -particle, $\Phi(r_{lj})$ is the interaction energy of two particles, \vec{p}_j the j -particle momentum and m its mass, $r_{lj} = |\vec{r}_l - \vec{r}_j|$ the distance between pair of interacting particles.

The function $\varrho(x^N; t)$ is symmetric regarding to inversion of phase variables of any pair of particles $x_l \leftrightarrow x_j$ and satisfies the normalization condition $\int d\Gamma_N \varrho(x^N; t) = 1$, $d\Gamma = (dx)^N / N!$, $dx = d\vec{p} d\vec{r}$.

In order to solve the Liouville equation (1) we will use the Zubarev nonequilibrium statistical operator method [2–4]. Within its framework we will be looking for solutions of the equation (1), which are independent of the initial conditions. The solutions will depend on time explicitly only, i.e. through the observable quantities

$$\int d\Gamma_N \hat{P}_n \varrho(x^N; t) = \langle \hat{P}_n \rangle^t \quad (3)$$

selected for the reduced description of nonequilibrium states of the system. In particular, for the description of the hydrodynamic state of the system of classical interacting particles the nonequilibrium averaged values of densities of number of particles $\langle \hat{n}(\vec{r}) \rangle^t$, their momentum

$\langle \hat{p}(\vec{r}) \rangle^t$ and total energy $\langle \hat{\varepsilon}(\vec{r}) \rangle^t$ can be chosen as the parameters of the reduced description [2–4]. They satisfy the respective conservation laws. In the case of kinetic description the one-particle $f_1(\vec{p}, \vec{r}; t)$ and two-particle $f_2(\vec{p}, \vec{r}, \vec{p}', \vec{r}'; t)$ distribution functions can serve as the reduced-description parameters. For the investigation of properties of magnetic and polar systems the averaged values of densities of magnetic $\langle \hat{m}(\vec{r}) \rangle^t$ and dipole $\langle \hat{d}(\vec{r}) \rangle^t$ moments can be used respectively besides the hydrodynamic variables.

When the basic parameters of the reduced description are defined, the solution $\varrho(x^N; t)$ can be presented by means of the nonequilibrium statistical operator method in the following general form [2–4]:

$$\varrho(x^N; t) = \varrho_{rel}(x^N; t) - \int_{-\infty}^t e^{\varepsilon(t'-t)} T(t, t') (1 - P_{rel}(t')) iL_N \varrho_{rel}(x^N; t') dt', \quad (4)$$

where $T(t, t') = \exp_+ \left\{ - \int_{t'}^t (1 - P_{rel}(t')) iL_N dt' \right\}$ is the evolution operator containing projection; \exp_+ the ordered exponential. $P_{rel}(t')$ is the generalized Kawasaki-Guntton projection operator whose structure depends on the form of the relevant statistical operator (distribution function) $\varrho_{rel}(x^N; t)$. The latter will be determined using maximum entropy principle for the Renyi entropy

$$S^R(\varrho) = \frac{1}{1-q} \ln \int d\Gamma_N \varrho^q(t). \quad (5)$$

The corresponding functional at fixed parameters of the reduced description with taking into account the normalization condition has the form

$$L^R(\varrho) = \frac{1}{1-q} \ln \int d\Gamma_N \varrho^q(t) - \alpha \int d\Gamma_N \varrho(t) - \sum_n F_n(t) \int d\Gamma_N \hat{P}_n \varrho(t), \quad (6)$$

$F_n(t)$ are the Lagrange multipliers. Equalizing its functional derivative to zero we obtain the relevant statistical operator corresponding to the Renyi entropy maximum:

$$\varrho_{rel}(t) = \frac{1}{Z_R} \left(1 - \frac{q-1}{q} \sum_n F_n(t) \delta \hat{P}_n \right)^{\frac{1}{q-1}}, \quad (7)$$

$$Z_R(t) = \int d\Gamma_N \left(1 - \frac{q-1}{q} \sum_n F_n(t) \delta \hat{P}_n \right)^{\frac{1}{q-1}}. \quad (8)$$

$Z_R(t)$ is the partition function of the relevant statistical operator, $\delta \hat{P}_n = \hat{P}_n - \langle \hat{P}_n \rangle^t$, and the parameter α in (6) is determined by the relation

$$\alpha = \frac{q}{1-q} - \sum_n F_n(t) \langle \hat{P}_n \rangle^t. \quad (9)$$

The Lagrange multipliers $F_n(t)$ in (7)-(9) are defined from the self-consistency conditions:

$$\langle \hat{P}_n \rangle^t = \langle \hat{P}_n \rangle_{rel}^t, \quad (10)$$

$\langle \dots \rangle_{rel}^t = \int d\Gamma_N \dots \varrho_{rel}(x^N; t)$. Since the relevant statistical operator is now known for the basic set of the reduced-description parameters, we can obtain the nonequilibrium statistical operator, establishing the structure of the projection operator:

$$\begin{aligned} P_{rel}(t)\varrho' &= \left(\varrho_{rel}(t) - \sum_n \frac{\delta \varrho_{rel}(t)}{\delta \langle \hat{P}_n \rangle^t} \langle \hat{P}_n \rangle^t \right) \int d\Gamma_N \varrho' \\ &+ \sum_n \frac{\delta \varrho_{rel}(t)}{\delta \langle \hat{P}_n \rangle^t} \int d\Gamma_N \hat{P}_n \varrho'. \end{aligned} \quad (11)$$

The variation derivative of the relevant statistical operator in (11) can be presented in the form:

$$\frac{\delta \varrho_{rel}(t)}{\delta \langle \hat{P}_m \rangle^t} = \varrho_{rel}(t) \delta \left[\frac{1}{q} \psi^{-1}(t) \left(F_m(t) - \sum_n \frac{\delta F_n(t)}{\delta \langle \hat{P}_m \rangle^t} \delta \hat{P}_n \right) \right], \quad (12)$$

where

$$\delta [\dots] = [\dots] - \langle [\dots] \rangle_{rel}^t, \quad (13)$$

and we use the notation

$$\psi(t) = 1 - \frac{q-1}{q} \sum_n F_n(t) \delta \hat{P}_n. \quad (14)$$

The Lagrange multipliers derivatives with regard to the reduced-description parameters $\delta F_n(t)/\delta \langle \hat{P}_m \rangle^t$ we calculate in the following way

$$\frac{\delta F_n(t)}{\delta \langle \hat{P}_m \rangle^t} = \left(\frac{\delta \langle \hat{P}_m \rangle^t}{\delta F_n(t)} \right)^{-1}. \quad (15)$$

It can be done in general case. Thus

$$\frac{\delta \langle \hat{P}_m \rangle^t}{\delta F_n(t)} = \frac{\delta \langle \hat{P}_m \rangle_{rel}^t}{\delta F_n(t)} = \int d\Gamma_N \hat{P}_m \frac{\delta \varrho_{rel}(t)}{\delta F_n(t)}. \quad (16)$$

After calculating $\delta \varrho_{rel}(t)/\delta F_n(t)$ in the right-hand side of the relation (16), we obtain the set of equations for desired derivatives

$$\frac{\delta \langle \hat{P}_m \rangle^t}{\delta F_n(t)} = \frac{1}{q} \langle \delta \hat{P}_m \psi^{-1}(t) \rangle_{rel}^t \sum_l \frac{\delta \langle \hat{P}_l \rangle^t}{\delta F_n(t)} - \frac{1}{q} \langle \delta \hat{P}_m \psi^{-1}(t) \delta \hat{P}_n \rangle_{rel}^t. \quad (17)$$

Its solution in the matrix form is

$$\frac{\delta\langle\hat{P}\rangle^t}{\delta F(t)} = - \left[I - \frac{1}{q} \langle \delta\hat{P}\psi^{-1}(t) \rangle_{rel}^t F \right]^{-1} \frac{1}{q} \langle \delta\hat{P}\psi^{-1}(t) \delta\hat{P} \rangle_{rel}^t = f(t), \quad (18)$$

where I is the unit matrix and

$$\frac{\delta\langle\hat{P}_m\rangle^t}{\delta F_n(t)} = \left(\frac{\delta\langle\hat{P}\rangle^t}{\delta F(t)} \right)_{mn} = f_{mn}(t). \quad (19)$$

Thus the functional derivative can be written in the form:

$$\frac{\delta\varrho_{rel}(t)}{\delta\langle\hat{P}_m\rangle^t} = \varrho_{rel}(t) \delta \left[\frac{1}{q} \psi^{-1}(t) \left(F_m(t) + \sum_n f_{mn}^{-1}(t) \delta\hat{P}_n \right) \right]. \quad (20)$$

Then the Kawasaki-Guntton projection operator has the following structure:

$$\begin{aligned} P_{rel}(t)\varrho' &= \varrho_{rel}(t) \int d\Gamma_N \{ \varrho' \} \\ &+ \sum_m \varrho_{rel}(t) \delta \left[\frac{1}{q} \psi^{-1}(t) \left(F_m(t) + \sum_n f_{mn}^{-1}(t) \delta\hat{P}_n \right) \right] \\ &\times \left(\int d\Gamma_N \{ \hat{P}_m \varrho' \} - \langle \hat{P}_m \rangle^t \int d\Gamma_N \{ \varrho' \} \right). \end{aligned} \quad (21)$$

It is further necessary to explore an action of the operators $P_{rel}(t)iL_N$ on the relevant statistical operator. Since

$$iL_N \varrho_{rel}(t) = -\varrho_{rel}(t) \frac{1}{q} \psi^{-1}(t) \sum_n F_n(t) \dot{\hat{P}}_n = A(t) \varrho_{rel}(t), \quad (22)$$

then

$$\begin{aligned} P_{rel}(t)iL_N \varrho_{rel}(t) &= P(t)A(t)\varrho_{rel}(t) = \int d\Gamma_N \{ A(t) \varrho_{rel}(t) \} \\ &+ \sum_m \varrho_{rel}(t) \delta \left[\frac{1}{q} \psi^{-1}(t) \left(F_m(t) + \sum_n f_{mn}^{-1}(t) \delta\hat{P}_n \right) \right] \\ &\times \left(\int d\Gamma_N \{ \hat{P}_m A(t) \varrho_{rel}(t) \} - \langle \hat{P}_m \rangle^t \int d\Gamma_N \{ A(t) \varrho_{rel}(t) \} \right), \end{aligned} \quad (23)$$

where

$$\int d\Gamma_N \{ \hat{P}_m A(t) \varrho_{rel}(t) \} - \langle \hat{P}_m \rangle^t \int d\Gamma_N \{ A(t) \varrho_{rel}(t) \} = \langle \delta\hat{P}_m A(t) \rangle_{rel}^t. \quad (24)$$

Thus $P_{rel}(t)iL_N \varrho_{rel}(t) = P_{rel}(t)A(t)\varrho_{rel}(t) = (P(t)A(t))\varrho_{rel}(t)$, where $P(t)$ is the projection operator which now acts on dynamic variables:

$$\begin{aligned} P(t) \dots &= \langle \dots \rangle_{rel}^t \\ &+ \sum_m \delta \left[\frac{1}{q} \psi^{-1}(t) \left(F_m(t) + \sum_n f_{mn}^{-1}(t) \delta\hat{P}_n \right) \right] \langle \dots \delta\hat{P}_m \rangle_{rel}^t. \end{aligned} \quad (25)$$

So far as

$$A(t) = -\frac{1}{q}\psi^{-1}(t) \sum_n F_n(t) \dot{\hat{P}}_n, \quad (26)$$

we have

$$\begin{aligned} P(t)A(t) &= -\frac{1}{q} \sum_n F_n(t) \langle \psi^{-1}(t) \dot{\hat{P}}_n \rangle_{rel}^t \\ &+ \sum_m \delta \left[\frac{1}{q} \psi^{-1}(t) \left(F_m(t) + \sum_l f_{ml}^{-1}(t) \delta \hat{P}_l \right) \right] \\ &\times \left\langle \left[-\frac{1}{q} \psi^{-1}(t) \sum_n F_n(t) \dot{\hat{P}}_n - \right. \right. \\ &\left. \left. \frac{1}{q} \sum_n F_n(t) \langle \psi^{-1}(t) \dot{\hat{P}}_n \rangle_{rel} \right] (\hat{P}_m - \langle \hat{P}_m \rangle_{rel}) \right\rangle_{rel}^t. \end{aligned} \quad (27)$$

Considering (22)-(27) we can present $(1 - P_{rel}(t))iL_N \varrho_{rel}(t)$ in the form:

$$\begin{aligned} (1 - P_{rel}(t))iL_N \varrho_{rel}(t) &= \\ &= (1 - P(t))iL_N \varrho_{rel}(t) = - \sum_n I_n(t) F_n(t) \varrho_{rel}(t), \end{aligned} \quad (28)$$

where

$$I_n(t) = (1 - P(t)) \frac{1}{q} \psi^{-1}(t) \dot{\hat{P}}_n \quad (29)$$

are the generalized flows. Taking into account (28) we can now write down an explicit expression for the nonequilibrium statistical operator

$$\begin{aligned} \varrho(x^N; t) &= \varrho_{rel}(x^N; t) \\ &+ \sum_n \int_{-\infty}^t e^{\varepsilon(t'-t)} T(t, t') I_n(t') F_n(t') \varrho_{rel}(x^N; t') dt'. \end{aligned} \quad (30)$$

It allows us to obtain the generalized transport equations for the reduced-description parameters. They can be presented in the form:

$$\frac{\partial}{\partial t} \langle \hat{P}_m \rangle^t = \langle \dot{\hat{P}}_m \rangle_{rel}^t + \sum_n \int_{-\infty}^t e^{\varepsilon(t'-t)} \varphi_{mn}(t, t') F_n(t') dt', \quad (31)$$

with the generalized transport kernels (memory functions)

$$\varphi_{mn}(t, t') = \int d\Gamma_N \{ \dot{\hat{P}}_m T(t, t') I_n(t') \varrho_{rel}(t') \} \quad (32)$$

which describe the dissipative processes in the system.

III. GENERALIZED TRANSPORT EQUATIONS FOR A CONSISTENT DESCRIPTION OF KINETICS AND HYDRODYNAMICS IN THE RENYI STATISTICS

For a consistent description of kinetic and hydrodynamic processes in classical (or quantum) systems of N particles interacting in the volume V the nonequilibrium one-particle distribution function $f_1(x; t) = \langle \hat{n}_1(x) \rangle^t$ and averaged value of potential energy of interaction $\varepsilon_{int}(\vec{r}; t) = \langle \hat{\varepsilon}_{int}(\vec{r}) \rangle^t$ are the basic parameters of the reduced description [44]. The last is defined through the two-particle distribution function $f_2(x, x'; t) = \langle \hat{n}_2(x, x') \rangle^t$. Here $\hat{n}_1(x)$ and $\hat{n}_2(x, x')$ are the phase densities of the microscopic distribution of particles, and $\hat{\varepsilon}_{int}(\vec{r}) = \frac{1}{2} \int d\vec{p} \int d\vec{p}' \int d\vec{r}' \Phi(|\vec{r} - \vec{r}'|) \hat{n}_2(x, x')$. In this case according to (7) we obtain the relevant statistical operator

$$\varrho_{rel}(t) = \frac{1}{Z_R} \left(1 - \frac{q-1}{q} \left\{ \int d\vec{r} \beta(\vec{r}; t) \delta \hat{\varepsilon}_{int}(\vec{r}; t) + \int dx a(x; t) \delta \hat{n}_1(x; t) \right\} \right)^{\frac{1}{q-1}}, \quad (33)$$

where

$$Z_R(t) = \int d\Gamma_N \left(1 - \frac{q-1}{q} \left\{ \int d\vec{r} \beta(\vec{r}; t) \delta \hat{\varepsilon}_{int}(\vec{r}; t) + \int dx a(x; t) \delta \hat{n}_1(x; t) \right\} \right)^{\frac{1}{q-1}} \quad (34)$$

is the partition function. The parameters $\beta(\vec{r}; t)$ and $a(x; t)$ are defined from the self-consistency conditions:

$$\langle \hat{\varepsilon}_{int}(\vec{r}) \rangle^t = \langle \hat{\varepsilon}_{int}(\vec{r}) \rangle_{rel}^t, \quad \langle \hat{n}_1(x) \rangle^t = \langle \hat{n}_1(x) \rangle_{rel}^t. \quad (35)$$

According to (4) and taking into consideration (33) we write down the nonequilibrium statistical operator of a consistent description of kinetic and hydrodynamic processes in the system:

$$\begin{aligned} \varrho(x^N; t) &= \varrho_{rel}(x^N; t) + \int_{-\infty}^t e^{\varepsilon(t'-t)} T(t, t') \\ &\times \left(\int d\vec{r}' \beta(\vec{r}'; t') I_{\varepsilon}^{int}(\vec{r}'; t') + \int dx' a(x'; t') I_n(x'; t') \right) \varrho_{rel}(x^N; t') dt'. \end{aligned} \quad (36)$$

Here

$$I_n(x'; t') = (1 - P(t')) \frac{1}{q} \psi^{-1}(t') i L_N \hat{n}_1(x') \quad (37)$$

$$I_{\varepsilon}^{int}(\vec{r}'; t') = (1 - P(t')) \frac{1}{q} \psi^{-1}(t') i L_N \hat{\varepsilon}_{int}(\vec{r}') \quad (38)$$

are the generalized flows. With the help of the nonequilibrium statistical operator obtained one can derive the generalized transport equations for the basic set of the reduced-description parameters according to (31):

$$\begin{aligned} \frac{\partial}{\partial t} \langle \hat{n}_1(x) \rangle^t &= \frac{1}{q} \int dx' \Phi_{n\dot{n}}(x, x'; t) a(x'; t) \\ &+ \frac{1}{q} \int d\vec{r} \Phi_{n\dot{\varepsilon}}(x, \vec{r}; t) \beta(\vec{r}; t) + \int dx' \int_{-\infty}^t e^{\varepsilon(t'-t)} \varphi_{nn}(x, x'; t, t') a(x'; t') dt' \\ &+ \int d\vec{r}' \int_{-\infty}^t e^{\varepsilon(t'-t)} \varphi_{n\varepsilon}(x, \vec{r}'; t, t') \beta(\vec{r}'; t') dt', \end{aligned} \quad (39)$$

$$\begin{aligned} \frac{\partial}{\partial t} \langle \hat{\varepsilon}_{int}(\vec{r}) \rangle^t &= \frac{1}{q} \int dx' \Phi_{\varepsilon\dot{n}}(\vec{r}, x'; t) a(x'; t) \\ &+ \frac{1}{q} \int d\vec{r}' \Phi_{\varepsilon\dot{\varepsilon}}(\vec{r}, \vec{r}'; t) \beta(\vec{r}'; t) + \int dx' \int_{-\infty}^t e^{\varepsilon(t'-t)} \varphi_{\varepsilon n}(\vec{r}, x'; t, t') a(x'; t') dt' \\ &+ \int d\vec{r}' \int_{-\infty}^t e^{\varepsilon(t'-t)} \varphi_{\varepsilon\varepsilon}(\vec{r}, \vec{r}'; t, t') \beta(\vec{r}'; t') dt'. \end{aligned} \quad (40)$$

The functions

$$\Phi_{n\dot{n}}(x, x'; t) = \int d\Gamma_N \hat{n}_1(x) \psi^{-1}(t) i L_N \hat{n}_1(x') \varrho_{rel}(x^N; t), \quad (41)$$

$$\Phi_{n\dot{\varepsilon}}(x, \vec{r}; t) = \int d\Gamma_N \hat{n}_1(x) \psi^{-1}(t) i L_N \hat{\varepsilon}_{int}(\vec{r}) \varrho_{rel}(x^N; t), \quad (42)$$

$$\Phi_{\varepsilon\dot{n}}(\vec{r}, x'; t) = \int d\Gamma_N \hat{\varepsilon}_{int}(\vec{r}) \psi^{-1}(t) i L_N \hat{n}_1(x') \varrho_{rel}(x^N; t), \quad (43)$$

$$\Phi_{\varepsilon\dot{\varepsilon}}(\vec{r}, \vec{r}'; t) = \int d\Gamma_N \hat{\varepsilon}_{int}(\vec{r}) \psi^{-1}(t) i L_N \hat{\varepsilon}_{int}(\vec{r}') \varrho_{rel}(x^N; t), \quad (44)$$

are the time correlation one obtained by means of the relevant distribution and which contain the function

$$\psi(t) = 1 - \frac{q-1}{q} \left(\int d\vec{r} \beta(\vec{r}; t) \delta \hat{\varepsilon}_{int}(\vec{r}; t) + \int dx a(x; t) \delta \hat{n}_1(x; t) \right), \quad (45)$$

At $q = 1$ $\psi(t) = 1$, we have the transition to the relevant Gibbs distribution when the nonequilibrium one-particle distribution function $f_1(x; t) = \langle \hat{n}_1(x) \rangle^t$ and averaged value of potential energy of interaction $\varepsilon_{int}(\vec{r}; t) = \langle \hat{\varepsilon}_{int}(\vec{r}) \rangle^t$ are the parameters of the reduced description [44]. The generalized transport kernels $\varphi_{nn}(x, x'; t, t')$, $\varphi_{n\varepsilon}(x, \vec{r}'; t, t')$, $\varphi_{\varepsilon n}(\vec{r}, x'; t, t')$, $\varphi_{\varepsilon\varepsilon}(\vec{r}, \vec{r}'; t, t')$ have the structure of (32) with the corresponding flows (37), (38).

For the case when the nonequilibrium one-particle distribution function $f_1(x; t) = \langle \hat{n}_1(x) \rangle^t$ is the single parameter of the reduced description (the contribution of nonequilibrium averaged potential energy of interaction is much smaller then the kinetic energy, e.g. the case

of rare gases) the set of transport equations (39)-(40) reduces to the generalized kinetic equation:

$$\begin{aligned} \frac{\partial}{\partial t} \langle \hat{n}_1(x) \rangle^t &= \frac{1}{q} \int dx' \Phi_{nn}(x, x'; t) a(x'; t) \\ &+ \int dx' \int_{-\infty}^t e^{\varepsilon(t'-t)} \varphi_{nn}(x, x'; t, t') a(x'; t') dt'. \end{aligned} \quad (46)$$

At $q = 1$ it transforms into the kinetic equation of [44] with the transport kernel calculated using relevant distribution function $\varrho_{rel}(t) = \prod_{j=1}^N \frac{f_1(x_j; t)}{e}$. In this case at $q = 1$ within the NSO method [2–4] the Liouville equation (1) should be solved with the boundary condition

$$\frac{\partial}{\partial t} \varrho(x^N; t) + iL_N \varrho(x^N; t) = -\varepsilon \left(\varrho(x^N; t) - \prod_{j=1}^N \frac{f_1(x_j; t)}{e} \right). \quad (47)$$

It corresponds to the Bogolyubov hypothesis of weakening of the correlations between particles. Integration of this equation with respect to phase variables $\int d\Gamma_{N-1}, \int d\Gamma_{N-2}, \dots, \int d\Gamma_{N-s}$ leads to the BBGKY hierarchy for the nonequilibrium particle distribution functions. The question about the BBGKY hierarchy when in the boundary condition (47) $\varrho_{rel}(t)$ is equal to (33) is interesting. With $q = 1$ $\varrho_{rel}(t)$ transforms into the Gibbs form and we obtain the BBGKY hierarchy with the modified boundary condition [3, 4, 44] which takes into account many-particle correlations. It allows one to obtain in the pair collision approximation the revised Enskog theory and the Enskog-Landau kinetic equations for the neutral and charged hard sphere system, respectively [3, 4, 44–46].

IV. CONCLUSIONS

For the nonequilibrium system of interacting particles within the framework of the Zubarev NSO method we obtained the nonequilibrium statistical operator $\varrho(t)$. It satisfies the Liouville equation with the boundary condition describing the relaxation of the NSO to the relevant statistical operator $\varrho_{rel}(t)$. The latter is constructed based on the maximum entropy principle for the Renyi entropy at fixed values of the reduced-description parameters $\langle \hat{P}_n \rangle^t$ with taking into account the normalization condition. By means of the NSO the generalized transport equations for the parameters of the reduced description $\langle \hat{P}_n \rangle^t$ are obtained with regard to Kawasaki-Gunton and Mori projection. Such an approach is applied to a consistent description of kinetic and hydrodynamic processes in the system

of classical interacting particles. As a result both the nonequilibrium statistical operator and the generalized transport equations are obtained, when the nonequilibrium one-particle distribution function $f_1(x;t) = \langle \hat{n}_1(x) \rangle^t$ along with the nonequilibrium averaged value of the potential energy of interaction $\varepsilon_{int}(\vec{r};t)$ are selected as the reduced-description parameters. At $q = 1$ the known results based on the Gibbs statistics are reproduced. Naturally, an interesting question about the investigation of time correlation functions and transport coefficients based on the NSO method within the Renyi statistics arises.

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